M.A./M.Sc. (Mathematics) Semester-III DR. SUNIL JE

Paper-XI (Functional Analysis) S.B. College

Unit 1:- Normed linear Space

(Defn.) Normed linear Space: Let E be a linear space over the field K of real or Complex numbers. A norm on E is a map ~~$\|\cdot\|:E$~~ $\|\cdot\|:E\to\mathbb{R}$ Satisfying the following Conditions:-

The image of an element $x\in E$ under the above map is denoted by $\|x\|$ and called the norm of x. It is a real no.

(N_1) $\|x\|\geq 0$ for every $x\in E$.

(N_2) $\|x\|=0$ iff $x=\theta$

(N_3) $\|\lambda x\|=|\lambda|\cdot\|x\|$ for every $\lambda\in K$ and $x\in E$.

(N_4) $\|x+y\|\leq\|x\|+\|y\|$ for all $x,y\in E$.

A linear space E in which a norm is defined is called a normed linear Space. A nls. is called a real or Complex nls according as the field K is the field ~~of~~ of real or Complex numbers.

Theorem: Q.No> Every normed linear space is a metric space.

Proof:- Let E be a nls. We define a map $d:E\times E\to\mathbb{R}$ by $d(x,y)=\|x-y\|$ for all $x,y\in E$.

then, Clearly, $d(x,y)=\|x-y\|\geq 0$ by (N_1).

Also, $d(x,y)=0$ iff $\|x-y\|=0$ iff $x-y=0$

i.e. iff $x=y$.

Again, $d(x, y) = \|x - y\| = \|(-1)(y - x)\|$

$= |-1| \cdot \|y - x\| = d(y, x)$ for all

$x, y \in E$, by (N_3).

Finally, for any $x, y, z \in E$,

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|, \text{ by } (N_4).$$

i.e. $d(x, z) \leq d(x, y) + d(y, z)$.

So, d becomes a metric on E, called the <u>metric generated by the norm of E</u>.

$\therefore (E, d)$ is a metric space.